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INTERVAL EQUITABLE EDGE COLORING OF THE GENERALIZED PETERSEN GRAPHS $P\ (n, 3), n > 7$

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ABSTRACT

Sudha et al. found the interval equitable edge coloring of the generalized Petersen graph $P(n, 2),\ n > 5$ and found its chromatic number to be 3. In this paper, we have extended our discussion of the interval equitable edge coloring of the generalized Petersen graph $P(n, 3),\ n > 7$ and establish that its chromatic number is also 3

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KEYWORDS: Color Class, Equitable Edge Coloring, Generalized Petersen Graph, Interval Edge Coloring, Interval Equitable Edge Coloring

INTRODUCTION

Mayer [1], introduced the equitable coloring of a graph and it is an assignment of colors to the vertices of the graph, in such a way that no two adjacent vertices have the same color and the number of vertices in any two color classes differ by at most one. Doro the [2] has discussed about the equitable coloring of complete multipartite graphs. Sudha et al. [3] have discussed about the equitable coloring of a prism of n-layers. Asratian et al. [4] introduced the concept of interval edge coloring. An edge coloring of a graph with colours $1, 2, \dots, t$ is an 'interval t-colouring' if all colours are used and the colours of edges incident to each vertex of a graph are distinct and they form an interval of integers. Kamalian [5] obtained the interval edge coloring of complete bipartite graphs and trees. Kamalian [6] also extended his result on cyclically interval edge coloring of simple cycles.

The generalized Petersen graph family was introduced by Coxeter and, these graphs were given their name by Mark Watkins [7] in 1969. Coxeter, has defined the generalized Petersen graph $P(n, k)$ for the natural numbers n and k, $n > 2k$, with the vertex set $\{u_i, v_i\}$, $1 \leq i \leq n$ and the edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+3}\}, 1 \leq i \leq n$ with subscripts reduced to modulo n. Coxeter [8], introduced the generalized Petersen graph in self-dual configurations and regular graphs. Babak Behsaz et al. [9] obtained the minimum vertex cover of generalized Petersen graphs.

Sudha et al. [10] introduced interval equitable edge coloring and also found the interval equitable edge coloring of Sudha gird of diamonds, a prism and the generalized Petersen graphs $P(n, 2),\ n > 5$.

In this paper, we have obtained the interval equitable edge coloring of the generalized Petersen graphs $P(n, 3)$, $n > 7$ and found its chromatic number for all $n > 7$, is 3

Definition 1

An edge-coloring of a graph is, the coloring of the edges of the graph with the minimum number of colors without any two adjacent edges having the same color.

Definition 2

In edge- coloring of a graph, the set of edges with the same color are said to be in the same color class.

In k- edge coloring of a graph, there are k color classes. The color classes are represented by C [1], C [2], ..., if $1, 2, ...$

Definition 3

An edge-coloring of a graph G with colors $1,2, \dots\ k$ is called an interval k- edge coloring if all the colors are used in such a way that the colors of the edges incident to any vertex of G, are distinct and are consecutive.

The smallest integer k for which the graph G is k-interval edge colored is known as the chromatic number of interval edge coloring and is denoted by $\chi_{ie}(G)$.

Definition 4

An interval equitable edge coloring, is an assignment of colors (positive integers) to the edges of the graph in such a way that

(i) No two adjacent edges have the same color

(ii) The set of colors defined on the edges incident to any vertex of the graph forms an interval and

(iii) The number of edges in any two color classes differs by at most one

Theorem 1

The generalized Petersen graph $P(n, 3), n > 7$ admits interval equitable edge coloring, and its interval equitable edge chromatic number is 3 for all $n > 7$.

Proof: Let the outer vertices and the inner vertices of the generalized Petersen graph $P(n, 3), n > 7$ be denoted by the sets $\{u_i\}$ and $\{v_i\}$, $1 \leq i \leq n$.

The set of edges of the generalized Petersen graph $P(n, 3)$ is given by $\{(u_i u_{i+1}), (v_i v_{i+3}), (u_i v_i) \,/\, 1 \leq i \leq n\}$ with the subscripts reduced to modulo n as shown in Figure 1.

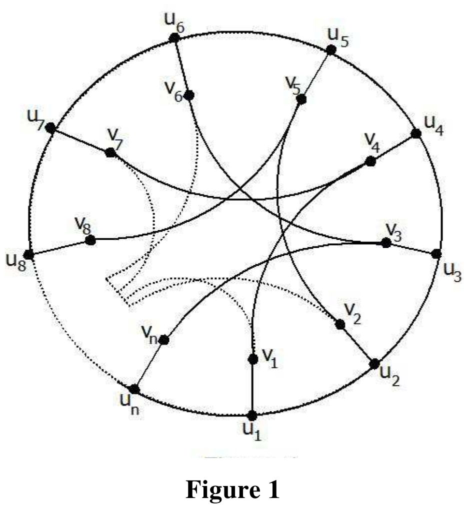


Figure 1

Define the function f, to be a mapping from the set of edges of the generalized Petersen graph $P(n,3), n > 7$, to the color set $\{1, 2, 3\}$ as follows:

There are six cases.

Case (i): Let $n \equiv 0\ (mod\ 6)$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$$

for $1 \le i \le n$.

The inner edges are colored as

$$f(v_i v_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$$

for $1 \le i \le n$.

The edges $\{u_i v_i\}$ are colored as

$$f\ (u_i v_i) = 1 \text{ for } 1 \le i \le n. \quad \textbf{(1)}$$

In this type of coloring, the Petersen graph $P(n,3), n > 7, n \equiv 0\ (mod\ 6)$ satisfies the definition of interval edge coloring. Since $|C[1]| = |C[2]| = |C[3]| = n$, $P(n,3), n > 7$, also satisfies the interval equitable edge coloring condition for $n \equiv 0\ (mod\ 6)$.

Case (ii): Let $n \equiv 1\ (mod\ 6)$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$$

for $2 < i < n - 1$,

$f(u_1 u_2) = 2,$

$f(u_2 u_3) = 1,$

$f(u_{n-1} u_n) = 1$

and $f(u_n u_1) = 3.$

The inner edges are colored as

$$f(v_i v_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$$

for $1 \le i < n - 2$,

$f(v_{n-2}v_1) = 3,$

$f(v_{n-1}v_2) = 1$

and $f(v_n v_3) = 1.$

The edges $\{u_i v_i\}$ are colored as

$f(u_i v_i) = 1$ for $3 < i < n - 1,$

$f(u_1 v_1) = 1,$

$f(u_2 v_2) = 3,$

$f(u_3 v_3) = 3,$

$f(u_{n-1} v_{n-1}) = 3$

and $f(u_n v_n) = 2.$ **(2)**

In this type of coloring the Petersen graph $P(n,3), n > 7$, $n \equiv 1 \ (mod\ 6)$ satisfies the definition of interval edge coloring. Since $|C[1]| = |C[2]| = |C[3]| = n$, $P(n,3), n > 7, which$ also satisfies the interval equitable edge coloring condition for $n \equiv 1 \ (mod\ 6)$.

Case (III): Let $n \equiv 2 \ (mod\ 6)$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$$

for $2 < \ i < n - 1,$

$f(u_1 u_2) = 2,$

$f(u_2 u_3) = 1,$

$f(u_{n-1} u_n) = 1$

and $f(u_n u_1) = 3.$

The inner edges are colored as

$$f(v_i v_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$$

for $1 \le i < n - 2,$

$f(v_{n-1}v_2) = 3,$

$f(v_{n-1}v_2) = 1$

and $f(v_n v_3) = 1.$

The edges $\{u_i v_i\}$ are colored as

$f(u_i v_i) = 1$ for $3 < i < n-1$,

$f(u_1 v_1) = 1$,

$f(u_2 v_2) = 3$,

$f(u_3 v_3) = 3$,

$f(u_{n-1} v_{n-1}) = 2$

And $f(u_n v_n) = 2$. **(3)**

In this type of coloring the Petersen graph$P(n,3), n > 7$, $n \equiv 2(mod\ 6)$ satisfies the definition of interval edge coloring. Since $|C[1]| = |C[2]| = |C[3]| = n$, $P(n,3), n > 7$ also satisfies the interval equitable edge coloring condition for$n \equiv 2\ (mod\ 6)$.

Case (iv): Let $n \equiv 3\ (mod\ 6)$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$$

for $2 < i < n-2$,

$f(u_1 u_2) = 2$,

$f(u_2 u_3) = 1$,

$f(u_{n-2} u_{n-1}) = 1$,

$f(u_{n-1} u_n) = 3$

and $f(u_n u_1) = 1$.

The inner edges are colored as

$$f(v_i v_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$$

for $1 \le i < n-2$,

$f(v_{n-2} v_1) = 1$,

$f(v_{n-1} v_2) = 1$

and $f(v_n v_3) = 1$.

The edges $\{u_i v_i\}$ are colored as

$f(u_i v_i) = 1$ for $3 < i < n-2$,

$f(u_1v_1) = 3,$

$f(u_2v_2) = 3,$

$f(u_3v_3) = 3,$

$f(u_{n-2}v_{n-2}) = 2,$

$f(u_{n-1}v_{n-1}) = 2$

and $f(u_nv_n) = 2.$ **(4)**

In this type of coloring the Petersen graph $P(n,3), n > 7$, $n \equiv 3\ (mod\ 6)$ satisfies the definition of interval edge coloring. Since $|C[1]| = |C[2]| = |C[3]| = n$, $P(n,3), n > 7$ also satisfies the interval equitable edge coloring condition for $n \equiv 3\ (mod\ 6)$.

Case (v): Let $\boldsymbol{n \equiv 4\ (mod\ 6)}$

The outer edges are colored as

$$f(u_iu_{i+1}) = \begin{cases} 2, i \equiv 0 (mod\ 2) \\ 3, i \equiv 1 (mod\ 2) \end{cases}$$

for $1 < i < n-2,$

$f(u_1u_2) = 1,$

$f(u_{n-2}u_{n-1}) = 1,$

$f(u_{n-1}u_n) = 3$

and $f(u_nu_1) = 2.$

The inner edges are colored as

$$f(v_iv_{i+3}) = \begin{cases} 2, i \equiv 1,2,3 (mod\ 6) \\ 3, i \equiv 0,4,5 (mod\ 6) \end{cases}$$

for $1 \leq i < n-2,$

$f(v_{n-2}v_1) = 1,$

$f(v_{n-1}v_2) = 1$

and $f(v_nv_3) = 3.$

The edges $\{u_iv_i\}$ are colored as

$f(u_iv_i) = 1$ for $2 < i < n-2,$

$f(u_1v_1) = 3,$

$f(u_2v_2) = 3,$

$f(u_{n-2}v_{n-2}) = 2,$

$f(u_{n-1}v_{n-1}) = 2$

and $f(u_n v_n) = 1.$ **(5)**

In this type of coloring the Petersen graph $P(n,3), n > 7$, $n \equiv 4\ (mod\ 6)$ satisfies the definition of interval edge coloring. Since, $|C[1]| = |C[2]| = |C[3]| = n$, $P(n,3), n > 7$ also satisfies the interval equitable edge coloring condition for $n \equiv 4\ (mod\ 6)$.

Case (vi): Let $n \equiv 5\ (mod\ 6)$

The outer edges are colored as

$$f(u_i u_{i+1}) = \begin{cases} 2, i \equiv 1 (mod\ 2) \\ 3, i \equiv 0 (mod\ 2) \end{cases}$$

for $2 < i < n-2$,

$f(u_1u_2) = 2,$

$f(u_2u_3) = 1,$

$f(u_{n-2}u_{n-1}) = 1,$

$f(u_{n-1}u_n) = 2$

and $f(u_n u_1) = 1.$

The inner edges are colored as

$$f(v_i v_{i+3}) = \begin{cases} 2, i \equiv 1,2,3 (mod\ 6) \\ 3, i \equiv 0,4,5 (mod\ 6) \end{cases}$$

for $1 \le i < n-2$,

$f(v_{n-2}v_1) = 1,$

$f(v_{n-1}v_2) = 1$

and $f(v_n v_3) = 1.$

The edges $\{u_i v_i\}$ are colored as

$f(u_i v_i) = 1$ for $3 < i < n-2$,

$f(u_1v_1) = 3,$

$f(u_2v_2) = 3,$

$f(u_3v_3) = 3,$

$f(u_{n-2}v_{n-2}) = 2,$

$f(u_{n-1}v_{n-1}) = 3$

and $f(u_nv_n) = 3.$ **(6)**

In this type of coloring the Petersen graph $P(n,3), n > 7$, $n \equiv 5\ (mod\ 6)$ satisfies the definition of interval edge coloring. Since $|C[1]| = |C[2]| = |C[3]| = n$, $P(n,3), n > 7$ also satisfies the interval equitable edge coloring condition for $n \equiv 5\ (mod\ 6)$.

For all the above six cases use the colors $\{1, 2, 3\}$. Hence, $\chi_{iee}\big(P(n,3)\big) = 3$ for $n > 7$.

Illustration

The illustration for the above six cases are given in Table 1.

Table 1: Illustration

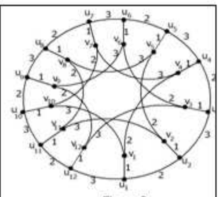
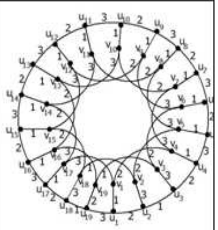
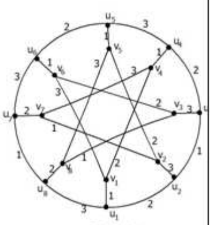
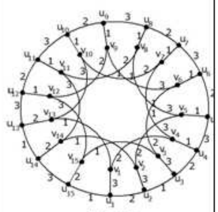
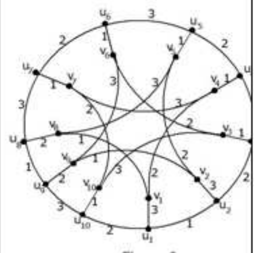
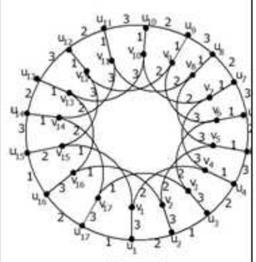
Case (i): $n \equiv 0\ (mod\ 6)$ Let $n = 12$ $P(12,3)$	for $1 \le i \le 12$, $f(u_iu_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$	for $1 \le i \le 12$, $f(v_iv_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$	for $1 \le i \le 12$, $f(u_iv_i) = 1$.	 Figure 2.
Case (ii): $n \equiv 1\ (mod\ 6)$ Let $n = 19$ $P(19,3)$	for $2 < i < 18$, $f(u_iu_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$ $f(u_1u_2) = 2$, $f(u_2u_3) = 1$, $f(u_{18}u_{19}) = 1$ and $f(u_{19}u_1) = 3$.	for $1 \le i < 17$, $f(v_iv_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$ $f(v_{17}v_1) = 3$, $f(v_{18}v_2) = 1$ and $f(v_{19}v_3) = 1$.	for $3 < i < 18$, $f(u_iv_i) = 1$ $f(u_1v_1) = 1, f(u_2v_2) = 3$, $f(u_3v_3) = 3, f(u_{18}v_{18}) = 3$ and $f(u_{19}v_{19}) = 2$.	 Figure 3.
Case (iii): $n \equiv 2\ (mod\ 6)$ Let $n = 8$ $P(8,3)$	for $2 < i < 7$, $f(u_iu_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$ $f(u_1u_2) = 2$, $f(u_2u_3) = 1$, $f(u_7u_8) = 1$ and $f(u_8u_1) = 3$.	for $1 \le i < 6, f(v_iv_{i+3}) =$ $\begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$ $f(v_6v_1) = 3$, $f(v_7v_2) = 1$ and $f(v_8v_3) = 1$.	for $3 < i < 7$, $f(u_iv_i) = 1$ $f(u_1v_1) = 1$, $f(u_2v_2) = 3, f(u_3v_3)$ $= 3, f(u_7v_7)$ $= 2$ and $f(u_8v_8) = 2$	 Figure 4.
Case (iv): $n \equiv 3\ (mod\ 6)$ Let $n = 15$ $P(15,3)$	for $2 < i < 13$, $f(u_iu_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$ $f(u_1u_2) = 2$, $f(u_2u_3) = 1$, $f(u_{13}u_{14}) = 1$, $f(u_{14}u_{15}) = 3$ and $f(u_{15}u_1) = 1$.	for $1 \le i < 13$ $f(v_iv_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$ $f(v_{13}v_1) = 1$, $f(v_{14}v_2) = 1$ and $f(v_{15}v_3) = 1$.	for $3 < i < 13$, $f(u_iv_i) = 1$, $f(u_1v_1) = 3$, $f(u_2v_2) = 3$, $f(u_3v_3) = 3$, $f(u_{13}v_{13}) = 2$, $f(u_{14}v_{14}) = 2$ and $f(u_{15}v_{15}) = 2$.	 Figure 5.

Table 1: Contd.,

Case (v): $n \equiv 4\ (mod\ 6)$ Let $n = 10$ $P(10, 2)$	for $1 < i < 8$, $f(u_i u_{i+1}) = \begin{cases} 2, i \equiv 0(mod\ 2) \\ 3, i \equiv 1(mod\ 2) \end{cases}$ $f(u_1 u_2) = 1$, $f(u_8 u_9) = 1$, $f(u_9 u_{10}) = 3$ and $f(u_{10} u_1) = 2$.	for $1 \leq i < 8$, $f(v_i v_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$ $f(v_8 v_1) = 1$, $f(v_9 v_2) = 1$ and $f(v_{10} v_3) = 3$.	for $2 < i < 8$, $f(u_i v_i) = 1$ $f(u_1 v_1) = 3$, $f(u_2 v_2) = 3$, $f(u_8 v_8) = 2$, $f(u_9 v_9) = 2$ and $f(u_{10} v_{10}) = 1$	 Figure 6.
Case (vi): $n \equiv 5\ (mod\ 6)$ Let $n = 17$ $P(17, 3)$	for $2 < i < 15$, $f(u_i u_{i+1}) = \begin{cases} 2, i \equiv 1(mod\ 2) \\ 3, i \equiv 0(mod\ 2) \end{cases}$ $f(u_1 u_2) = 2$, $f(u_2 u_3) = 1$, $f(u_{15} u_{16}) = 1$, $f(u_{16} u_{17}) = 2$ and $f(u_{17} u_1) = 1$.	for $1 \leq i < 15$, $f(v_i v_{i+3}) = \begin{cases} 2, i \equiv 1,2,3(mod\ 6) \\ 3, i \equiv 0,4,5(mod\ 6) \end{cases}$ $f(v_{15} v_1) = 1$, $f(v_{16} v_2) = 1$ and $f(v_{17} v_3) = 1$.	for $3 < i < 15$, $f(u_i v_i) = 1$ $f(u_1 v_1) = 3$, $f(u_2 v_2) = 3$, $f(u_3 v_3) = 3$, $f(u_{15} v_{15}) = 2$, $f(u_{16} v_{16}) = 3$ and $f(u_{17} v_{17}) = 3$.	 Figure 7.

CONCLUSIONS

We have found that, the chromatic number of the interval equitable edge coloring of the generalized Petersen graph $P\ (n, 3)$ for all $n > 7$, is 3.

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